COUPLED WAVE EQUATIONS IN MAGNETO-IONIC THEORY

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ABSTRACT. It has been shown that the coupled wave-equations deduced by Saha, Banerjea and Guha (1951) are *incorrectly* labelled. The wave-polarization of the radio-wave and the relations between the electric and the magnetic fields have been deduced from the coupled wave-equations.

INTRODUCTION

By rotating the co-ordinate system through a complex angle ϕ Saha, Banerjea and Guha (1951) obtained the following wave-equations for the radio-wave propagation vertically at any latitude through a horizontally stratified ionosphere :

$$\frac{d^2V}{du^2}+(q_0^2-\dot{\phi}^2)V = 2\dot{\phi}\dot{W}+\ddot{\phi}W \text{ (for the 0-mode).} \quad \ldots (1)$$

$$\frac{d^2W}{du^2}+(q_x^2-\dot{\phi}^2)W = -2\dot{\phi}\dot{V}-\ddot{\phi}V \text{ (for the } X\text{-mode)} \quad \ldots (1.1)$$

where V and W are the propagation vectors defined by :

$$V = E_x \cos\phi+jE_y \sin\phi = \frac{E_x+j\rho_1 E_y}{\sqrt{1+\rho_1^2}} \quad \ldots (2)$$

$$W = -E_x \sin\phi+jE_y \cos\phi = \frac{E_x+j\rho_2 E_y}{\sqrt{1+\rho_2^2}} \quad \ldots (3)$$

and $u = \frac{2\pi}{\lambda} Z$, λ being the wavelength in free-space.

Here E_x and E_y are the normal and the abnormal components of the electric vector, the direction of propagation being along the Z-direction (See Fig. 2). The symbols in (1) and (1.1) have the following significance :

$$q_0^2 = K_1\cos^2\phi+K_2\sin^2\phi-2L\sin\phi\cos\phi = 1-\frac{r}{\beta'+\rho_1\omega_z} \quad \ldots (4)$$

$$q_x^2 = K_1\sin^2\phi+K_2\cos^2\phi+2L\sin\phi\cos\phi = 1-\frac{r}{\beta'+\rho_2\omega_z} \quad \ldots (5)$$

$$\rho_1 = G-\sqrt{1+G^2}\ , \quad \rho_2 = G+\sqrt{1+G^2} \quad \ldots (6)$$

$$\left.\begin{aligned}
G &= -\cot 2\phi = \frac{K_1-K_2}{2L} = \frac{\omega_x^2}{2\omega_z(r-\beta')} \\
K_1 &= 1-\frac{\beta'^2-r\beta'-\omega_x^2}{c'}r,\; K_2 = 1-\frac{r(\beta'^2-r\beta')}{c'},\; L = \frac{r(r-\beta')w_z}{c'} \\
c' &= \beta'(\beta'^2-\omega^2)-r(\beta'^2-\omega_z^2),\quad \omega_x = \omega\sin\theta,\quad \omega_z = \omega\cos\theta \\
\beta' &= 1-j\frac{\nu}{p},\; \omega = \frac{p_H}{p},\; p_H = \frac{eH}{mc},\; r = \frac{p_0^2}{p^2} = \frac{4\pi Ne^2}{mp^2} \\
\dot{V} &= \frac{dV}{du},\; \dot{W} = \frac{dW}{du},\; \dot{\phi} = \frac{d\rho/du}{1+\rho^2}
\end{aligned}\right\} \quad \ldots (7)$$

COMPLEX REFRACTIVE INDEX

The well-known Appleton-Hartree formula (1927, 1929) for the square of complex refractive index q is given by

$$q^2 = 1+\frac{1}{(\alpha+j\beta)-\dfrac{\gamma_T^2}{2(1+\alpha+j\beta)}\pm\sqrt{\dfrac{\gamma_T^4}{4(1+\alpha+j\beta)^2}+\gamma_L^2}} \quad \ldots (8)$$

The upper positive sign before the radical in eq. (8) refers to the extraordinary mode and the lower negative sign to the ordinary mode.

The notations in (8) are :

$$\alpha = -p^2/p_0^2,\; p_0^2 = 4\pi Ne^2/m,\; \beta = p\nu/p_0^2$$

$$\gamma = pp_H/p_0^2,\; p_H = eH/mc,\; \gamma_L = \gamma\cos\theta,\; \gamma_T = \gamma\sin\theta$$

where

ν = electron collisional frequency

H = intensity of the earth's magnetic field

e, m = charge and mass of an electron

c = velocity of light in vacuum,

p = angular frequency of the wave,

N = electron number density

and θ = angle between the direction of propagation of the radio-wave and the positive direction of the earth's magnetic field.

In terms of the U.R.S.I. notations, Eq. (8) corresponds to :

$$q^2 = 1-\frac{X}{1-jZ-\dfrac{Y_T^2}{2(1-X-jZ)}\pm\sqrt{\left[\dfrac{Y_T^4}{4(1-X-jZ)^2}+Y_L^2\right]}} \quad \ldots (8.1)$$

The notations in Eq. (8.1) are:

$$X = \frac{4\pi Ne^2}{m\omega^2}, \; Y_{L,T} = \frac{eH_{L,T}}{\omega mc}, \; Z = \frac{\nu}{\omega}$$

The angular frequency ω of radio-wave is the same as p in the old notation. Associating the *plus* sign before the radical in Eq. (8.1) with the ordinary mode and the *minus* sign with the *extraordinary* mode (Ratcliffe, 1962), it can be easily shown that the minus sign before the radical in Eq. (8) corresponds to the ordinary mode and the plus sign to the extraordinary mode, so that, following the old notations we get for the 0-mode:

$$q_0^2 = 1 + \frac{1}{\alpha + j\beta - \frac{\gamma_T^2}{2(1+\alpha+j\beta)} - \sqrt{\left[\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2} + \gamma_L^2\right]}} \qquad \ldots \quad (8.2)$$

and for the X-mode:

$$q_x^2 = 1 + \frac{1}{\alpha + j\beta - \frac{\gamma_T^2}{2(1+\alpha+j\beta)} + \sqrt{\left[\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2} + \gamma_L^2\right]}} \qquad \ldots \quad (8.3)$$

We next compare these two formulae (8.2), (8.3) with equations (4) and (5) deduced from the coupled wave-equations of Saha, Banerjea and Guha (1951). From (7) it can be easily shown:

$$G = \frac{\gamma_T^2}{2\gamma_L(1+\alpha+j\beta)} \qquad \ldots \quad (7.1)$$

Since $\qquad \omega_z = \frac{p_H}{p} \cos\theta$, we get:

$$\rho_1\omega_z = (G - \sqrt{1+G^2})\frac{p_H}{p}\cos\theta$$

$$= \frac{p_0^2}{p^2}\left[\frac{\gamma_T^2}{2(1+\alpha+j\beta)} - \sqrt{\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2} + \gamma_L^2}\right]$$

Hence putting the value of $\rho_1\omega_z$ in Eq. (4), we have

$$q_0^{\,2} = 1 + \frac{1}{\alpha + j\beta - \frac{\gamma_T^2}{2(1+\alpha+j\beta)} + \sqrt{\frac{\gamma_T}{4(1+\alpha+j\beta)^2} + \gamma_L^{2}}}$$

It is seen that the last expression for the square of complex refractive index agrees with Eq. (8.3). Similarly, it can be shown that Eq. (5) agrees with (8.2). Hence Eqs. (4) and (5) should be interchanged as follows :

$$q_x^2 = 1 - \frac{r}{\beta' + \rho_1 \omega_z}$$

and

$$q_0^2 = 1 - \frac{r}{\beta' + \rho_2 \omega_z}$$

COUPLED WAVE-EQUATIONS AND PROPAGATION VECTORS

Saha *et al* (1951) deduced the coupled wave-equations (1) and (1.1) by starting from the following equations :

$$\frac{d^2 E_x}{du^2} + K_1 E_x - jLE_y = 0 \qquad \ldots \quad (9)$$

$$\frac{d^2 E_y}{du^2} + K_2 E_y + jLE_x = 0 \qquad \ldots \quad (9.1)$$

and putting :

$$\begin{pmatrix} E_x \\ jE_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix} \qquad \ldots \quad (10)$$

Using Eqs. (9), (9.1) and (10), it can be easily shown :

$$V'' + [K_1 \cos^2\phi + K_2 \sin^2\phi - 2L \sin\phi \cos\phi) V$$

$$-[(K_1 - K_2) \sin\phi \cos\phi + L(\cos^2\phi - \sin^2\phi)] W = 0 \qquad \ldots \quad (1.2)$$

$$W'' + [K_1 \sin^2\phi + K_2 \cos^2\phi + 2L \cos\phi \sin\phi] W$$

$$-[(K_1 - K_2) \sin\phi \cos\phi + L(\cos^2\phi - \sin^2\phi)] V = 0 \qquad \ldots \quad (1.3)$$

where

$$V'' = \ddot{V} - 2\dot{\phi}\dot{W} - \ddot{\phi} W - \dot{\phi}^2 V \qquad \ldots \quad (11)$$

$$W'' = \ddot{W} + 2\dot{\phi}\dot{V} + \ddot{\phi} V - \dot{\phi}^2 W \qquad \ldots \quad (11.1)$$

The coefficients of the cross-terms in Eqs. (1.2) and (1.3) may be made to disappear by writing $\tan\phi = G \pm 1 + G^2$ as shown by Saha *et al.* Hence there are two values of $\tan\phi$ given by $G + \sqrt{1+G^2}$ and $G - \sqrt{1+G^2}$.

Let us write :

$$\tan \phi_1 = G-\sqrt{1+G^2} = \rho_1 \quad \dots \quad (12)$$

$$\tan \phi_2 = G+\sqrt{1+G^2} = \rho_2 \quad \dots \quad (12.1)$$

We next rotate the co-ordinate system through a complex angle $\phi_1 = \tan^{-1}\rho_1$. The equations (1.2) and (1.3) are then reduced to

$$V''_1+[K_1 \cos^2 \phi_1+K_2 \sin^2 \phi_1-2L \sin \phi_1 \cos \phi_1]V_1 = 0 \quad \dots \quad (1.4)$$

$$W''_1+[K_1 \sin^2 \phi_1+K_2 \cos^2 \phi_1+2L \sin \phi \cos \phi_1]\, W_1 = 0 \quad \dots \quad (1.5)$$

where

$$V_1'' = \ddot{V}_1-2\dot{\phi}_1\dot{W}_1-\ddot{\phi}_1 W_1-\dot{\phi}_1^2 V_1 \quad \dots \quad (11.2)$$

$$W''_1 = \ddot{W}_1+2\dot{\phi}_1\dot{V}_1+\ddot{\phi}_1 V_1-\dot{\phi}^2 W_1 \quad \dots \quad (11.3)$$

$$V_1 = E_x \cos \phi_1+jE_y \sin \phi_1 \quad \dots \quad (2.1)$$

$$W_1 = -E_x \sin \phi_1+jE_y \cos \phi_1 \quad \dots \quad (3.1)$$

If we rotate the co-ordinate system through a complex angle $\phi_2 = \tan^{-1}\rho_2$ then the equations (1.2) and (1.3) are reduced to

$$V''_2+[K_1 \cos^2 \phi_2+K_2 \sin^2 \phi_2-2L \sin \phi_2 \cos \phi_2]V_2 = 0 \quad \dots \quad (1.6)$$

$$W''_2+[K_1 \sin^2 \phi_2+K_2 \cos^2 \phi_2+2L \sin \phi_2 \cos \phi_2]W_2 = 0 \quad \dots \quad (1.7)$$

where

$$V''_2 = \ddot{V}_2-2\dot{\phi}_2\dot{W}_2-\ddot{\phi}_2 W_2-\dot{\phi}_2^2 V_2 \quad (11.4)$$

$$W''_2 = \ddot{W}_2+2\dot{\phi}_2\dot{V}_2+\ddot{\phi}_2 V_2-\dot{\phi}_2^2 W_2 \quad \dots \quad (11.5)$$

$$V_2 = E_x \cos \phi_2+jE_y \sin \phi_2 \quad \dots \quad (2.2)$$

$$W_2 = -E_x \sin \phi_2+jE_y \cos \phi_2 \quad \dots \quad (3.2)$$

It is shown in the Appendix that the following relations hold good :

$$K_1 \cos^2 \phi_1+K_2 \sin^2 \phi_1-2L \sin \phi_1 \cos \phi_1 = 1-\frac{r}{\beta'+\rho_1\omega_z} = q_x^2 \quad \dots \quad (4.1)$$

$$K_1 \cos^2 \phi_2+K_2 \sin^2 \phi_2-2L \sin \phi_2 \cos \phi_2 = 1-\frac{r}{\beta'+\rho_2\omega_z} = q_0^3 \quad \dots \quad (5.1)$$

$$K_1 \sin^2 \phi_1+K_2 \cos^2 \phi_1+2L \sin \phi_1 \cos \phi_1 = 1-\frac{r}{\beta'+\rho_2\omega_z} = q_0^2 \quad \dots \quad (5.2)$$

$$K_1 \sin^2 \phi_2 + K_2 \cos^2 \phi_2 + 2L \sin \phi_2 \cos \phi_2 = 1 - \frac{r}{\beta' + \rho_1 \omega_z} = q_x^2 \quad \ldots \quad (4.2)$$

Hence equations (1.4) and (1.5) should be called the coupled wave-equations for the X-and 0-modes respectively and can be written as :

$$\ddot{V}_1 + (q_x^2 - \dot{\phi}_1^2) V_1 = 2\dot{\phi}_1 \dot{W}_1 + \ddot{\phi}_1 W_1 \text{ (for the } X\text{-mode)} \quad \ldots \quad (1.8)$$

$$\ddot{W}_1 + (q_0^2 - \dot{\phi}_2^2) W_1 = -2\dot{\phi}_1 \dot{V}_1 - \ddot{\phi}_1 V_1 \text{ (for the 0—mode)} \quad \ldots \quad (1.9)$$

where,

$$V_1 = \frac{E_x + j\rho_1 E_y}{\sqrt{1+\rho_1^2}} \text{ (for the } X\text{—mode)} \quad \ldots \quad (2.3)$$

$$W_1 = \frac{-\rho_1 E_x + jE_y}{\sqrt{1+\rho_1^2}} \text{ (for the 0- mode)} \quad \ldots \quad (3.3)$$

Similarly Eqs. (1.6) and (1.7) can be rewritten as :

$$\ddot{V}_2 + (q_0^2 - \dot{\phi}_2^2) V_2 = 2\dot{\phi}_2 \dot{W}_2 + \ddot{\phi}_2 W_2 \text{ (for the 0-mode)} \quad \ldots \quad (1.10)$$

$$\ddot{W}_2 + (q_x^2 - \dot{\phi}_2^2) W_2 = -2\dot{\phi}_2 \dot{V}_2 - \ddot{\phi}_2 V_2 \text{ (for the } X\text{-mode)} \quad \ldots \quad (1.11)$$

where

$$V_2 = \frac{E_x + j\rho_2 E_y}{\sqrt{1+\rho_2^2}} \text{ (for the 0-mode)} \quad \ldots \quad (2.4)$$

$$W_2 = \frac{-\rho_2 E_x + jE_y}{\sqrt{1+\rho_2^2}} \text{ (for the } X\text{-mode)} \quad \ldots \quad (3.4)$$

It has been shown in the Appendix

$$W_1 = V_2, \; W_2 = -V_1 \quad \dot{\phi}_1 = \dot{\phi}_2 = \dot{\phi}\text{(say)} \quad \ldots \quad (13)$$

Hence, using (13), the equations (1.8) (1.9), (1.10) and (1.11) can be combined into two coupled epuations : –

$$\ddot{V}_x + (q_x^2 - \dot{\phi}^2) V_x = 2\dot{\phi} \dot{V}_0 + \ddot{\phi} V_0 \text{ (for the } X\text{-mode)} \quad \ldots \quad (1.12)$$

$$\ddot{V}_0 + (q_0^2 - \dot{\phi}^2) V_0 = -2\dot{\phi} \dot{V}_x - \ddot{\phi} V_x \text{ (for the 0-mode)} \quad \ldots \quad (1.13)$$

where,

$$V_x = V_1 = -W_2 = \frac{E_x^{(x)} + j\rho_1 E_y^{(x)}}{\sqrt{1+\rho_1^2}} = \text{Propagation vector for the } X\text{-mode} \quad (2.5)$$

$$V_0 = V_2 = W_1 = \frac{E_x^{(0)} + j\rho_2 E_y^{(0)}}{\sqrt{1+\rho_2^2}} = \text{Propagation vector for the 0-mode} \quad \ldots \quad (3.5)$$

$$\dot{\phi}_1 = \frac{d\rho_1/du}{1+\rho_1^2} = \frac{d\rho_2/du}{1+\rho_2^2} = \dot{\phi}_2 = \dot{\phi}$$

$$q_x^2 = 1 - \frac{r}{\beta' + \rho_1 \omega_z}, \quad q_0^2 = 1 - \frac{r}{\beta' + \rho_2 \omega_z}$$

Thus it is evident that the equations (1) and (1.1) were *incorrectly* labelled as the coupled wave-equations for the 0-mode and the X-mode respectively. It is shown in the next section that the equations (1) and (1.1) lose all their significance, if they are called the coupled wave-equations for the 0- and X-modes respectively.

WAVE-POLARIZATION

We start from the relations :

$$\begin{aligned} D_x &= q^2 E_x \\ D_y &= q^2 E_y \end{aligned} \quad \ldots \quad (14)$$

The following expressions for the displacement vector were deduced by Saha *et al* (1951).

$$\begin{aligned} D_x &= K_1 E_x - jLE_y \\ D_y &= K_2 E_y + jLE_x \end{aligned} \quad \ldots \quad (15)$$

From (14) and (15)

$$\frac{E_x}{E_y} = \frac{K_2 - q^2 - jL}{q^2 - K_1 - jL} = \frac{q^2 - jL - K_2}{q^2 - K_1 + jL} \quad \ldots \quad (16)$$

Hence from (16)

$$q^2 = \frac{(K_1+K_2) + \sqrt{(K_1+K_2)^2 - 4(K_1K_2 - L^2)}}{2} \quad \ldots \quad (17)$$

From (16) and (17)

$$\frac{E_y - E_x}{E_y + E_x} = \pm \frac{\sqrt{(K_2 - K_1)^2 + 4L^2}}{(K_2 - K_1) - 2jL} \quad \ldots \quad (16.1)$$

Taking the positive sign of Ea. (16.1)

$$\frac{E_y - E_x}{E_y + E_x} = \sqrt{\frac{(K_2 - K_1) + 2jL}{(K_2 - K_1) - 2jL}} \quad \ldots \quad (16.2)$$

Since $G=(K_1-K_2)/2L$ we get :

$$\frac{E_y-E_x}{E_y+E_x}=\sqrt{\frac{G-j}{G+j}} \qquad \dots \quad (16.3)$$

Putting $G=A\cos\psi$, $1=A\sin\psi$ where $A=\sqrt{1+G^2}$ we get from (16.3)

$$\frac{E_x}{E_y}=\tanh j\psi/2 \qquad \dots \quad (16.4)$$

Since $\tan\psi=\frac{1}{G}=-\tan 2\phi$, we have

$$\frac{E_x}{E_y}=-j\tan\phi=-j\rho \qquad \dots \quad (16.5)$$

The same Eq. (16.5) can be deduced by using the negative sign of Eq.(16.1).

There are two different values of ρ, viz. $\rho_1=G-\sqrt{1+G^2}$ and $\rho_2=G+\sqrt{1+G^2}$ and we have

$$\frac{E_x}{E_y}=-j\rho_1 \qquad \dots \quad (16.6)$$

$$\frac{E_x}{E_y}=-j\rho_2 \qquad \dots \quad (16.7)$$

Let us now compare these equations (16.6) and (16.7) with the well-known Appleton-Hartree formula (1927-29) for the wave polarization in order to associate eqs. (16.6) and (16.7) with the so-called *0*- and *X*- modes.

Using the right-handed co-ordinate system (Fig. 1) the Appleton-Hartree formula for the wave-polarization in terms of magnetic vector components can be written as :

$$\frac{H_z}{H_y}=\frac{j}{\gamma_L}\left[-\frac{\gamma_T^2}{2(1+\alpha+j\beta)}\pm\sqrt{\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2}+\gamma_L^2}\right] \qquad \dots \quad (18)$$

where the direction of propagation of the radio wave is along the X-axis. In Eqs. (18), (16.6) and (16.7) the sign of the charge has not been taken into consideration.

In deriving the equations (16.6) and (16.7) the co-ordinate system of Fig. 2 has been used. When eq. (18) is referred to the co-ordinate system of Fig. 2, we have :

$$\frac{H_x}{H_y}=-\frac{j}{\gamma_L}\left[-\frac{\gamma_T^2}{2(1+\alpha+j\beta)}+\sqrt{\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2}+\gamma_L^2}\right] \qquad \dots \quad (18.1)$$

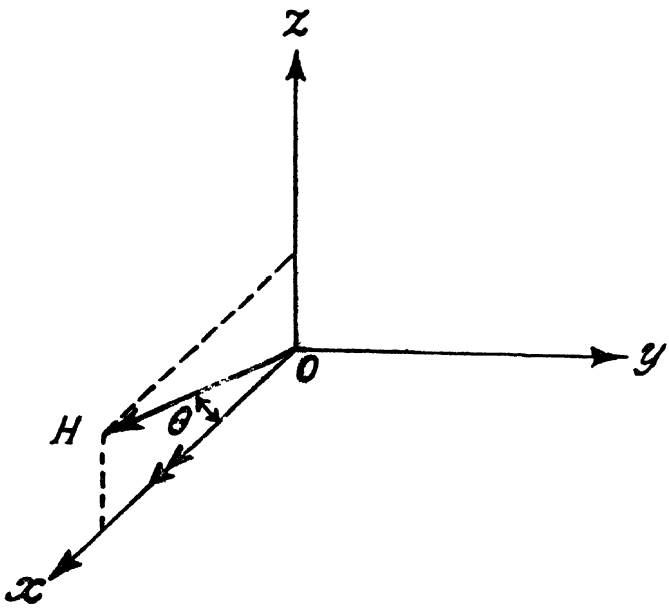

Fig. 1. Co-ordinate system (Saha *et al*).
OH → direction of the earth's magnetic field.
OZ → direction of the radio wave propagation.

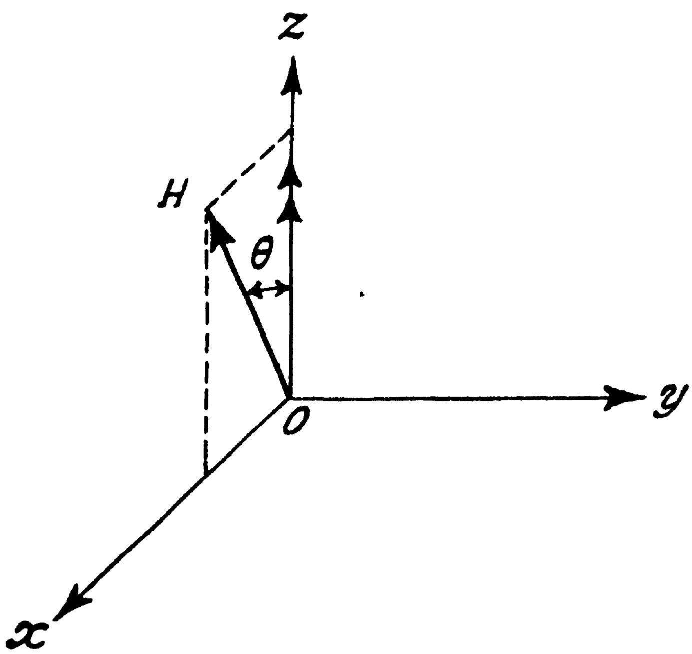

Fig. 2. Co-ordinate system (Appleton)
OH → direction of the earth's magnetic field.
OX → direction of the radio wave propagation.

where the direction of propagation of radio-wave is along Z-axis. Hence for the 0-mode :

$$\left(\frac{H_x}{H_y}\right)_0 = -\frac{j}{\gamma_L}\left[-\frac{\gamma_T^2}{2(1+\alpha+j\beta)} - \sqrt{\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2}+\gamma_L^2}\right] \qquad \dots \quad (18.2)$$

and for the X-mode :

$$\left(\frac{H_x}{H_y}\right)_x = -\frac{j}{\gamma_L}\left[-\frac{\gamma_T^2}{2(1+\alpha+j\beta)} + \sqrt{\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2}+\gamma_L^2}\right] \qquad \dots \quad (18.3)$$

Since $E_x/E_y = -H_y/H_x$ we get from (18.2) and (18.3)

$$\left(\frac{E_x}{E_y}\right)_0 = -\frac{j}{\gamma_L}\left[\frac{\gamma_T^2}{2(1+\alpha+j\beta)} - \sqrt{\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2}+\gamma_L^2}\right] \qquad \dots \quad (18.4)$$

$$\left(\frac{E_x}{E_y}\right)_x = -\frac{j}{\gamma_L}\left[\frac{\gamma_T^2}{2(1+\alpha+j\beta)} + \sqrt{\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2}+\gamma_L^2}\right] \qquad \dots \quad (18.5)$$

From (7.1) and (12)

$$-j\rho_1 = -\frac{j}{\gamma_L}\left[\frac{\gamma_T^2}{2(1+\alpha+j\beta)} - \sqrt{\frac{\gamma_T^4}{4(1+\alpha+j\beta)^2}+\gamma_L^2}\right]$$

Since the above expression for $-j\rho_1$ agrees with (18.4) we have for the 0-mode :

$$\left(\frac{E_x}{E_y}\right)_0 = -j\rho_1 \qquad \dots \quad (16.8)$$

Similarly for the X-node :

$$\left(\frac{E_x}{E_y}\right)_x = -j\rho_2 \qquad \dots \quad (16.9)$$

In view of Eqs. (16.8) and (16.9) it is seen from (2) and (3) that V and W are reduced to zero; hence the equations (1) and (1.1) lose all their significance, if these wave equations (1) and (1.1) are associated with the 0- and X-mode respectively.

RELATION BETWEEN THE ELECTRIC AND MAGNETIC FIELDS

We start from the eqs. of propagation of the magnetic vector (1947) :

$$\frac{d^2H_x}{du^2} + K_2H_x - jLH_y = 0$$

$$\frac{d^2H_y}{du^2} + K_1H_y + jLH_x = 0 \qquad \dots \quad (19)$$

and rotate the co-ordinate system (Fig. 2) through a complex angle ϕ' and put

$$\begin{pmatrix} jH_y \\ H_x \end{pmatrix} = \begin{pmatrix} \cos\phi' & -\sin\phi' \\ \sin\phi' & \cos\phi' \end{pmatrix} \begin{pmatrix} V_H \\ W_H \end{pmatrix} \qquad \dots \quad (20)$$

Using (19) and (20) it can be easily shown :

$$\ddot{V}_H+[K_1\cos^2\phi'+K_2\sin^2\phi'-2L\sin\phi'\cos\phi'-\dot{\phi}'^2]V_H$$

$$-[L(\cos^2\phi'-\sin^2\phi')+(K_1-K_2)\sin\phi'\cos\phi']W_H = 2\dot{\phi}'\dot{W}_H+\ddot{\phi}'W_H \quad \dots \quad (19.1)$$

and

$$\ddot{W}_H+[K_1\sin^2\phi'+K_2\cos^2\phi'+2L\sin\phi'\cos\phi'-\dot{\phi}'^2]W_H$$

$$-[(K_1-K_2)\cos\phi'\sin\phi'+L(\cos^2\phi'-\sin^2\phi')]V_H = -2\dot{\phi}'\dot{V}_H-\ddot{\phi}'V_H \quad \dots \quad (19.2)$$

Putting $\dfrac{K_1-K_2}{2L} = G = -\cot 2\phi'$ Eqs. (19.1) and (19.2) are reduced to

$$\ddot{V}_H+[K_1\cos^2\phi'+K_2\sin^2\phi'-2L\sin\phi'\cos\phi'-\dot{\phi}'^2]V_H$$

$$= 2\dot{\phi}'\dot{W}_H+\ddot{\phi}'W_H \qquad \dots \quad (19.3)$$

and,

$$\ddot{W}_H+[K_1\sin^2\phi'+K_2\cos^2\phi'+2L\sin\phi'\cos\phi'-\dot{\phi}'^2]W_H$$

$$=-2\dot{\phi}'\dot{V}_H-\ddot{\phi}'W_H \qquad \dots \quad (19.4)$$

where,

$$V_H = jH_y\cos\phi'+H_x\sin\phi' \qquad \dots \quad (2.6)$$

$$W_H = -jH_y\sin\phi'+H_x\cos\phi' \qquad \dots \quad (3.6)$$

Putting $\dfrac{K_1-K_2}{2L} = G = -\cot 2\phi$ Eqs (1.2) and (1.3) can be written as :

$$\ddot{V}+[K_1\cos^2\phi+K_2\sin^2\phi-2L\sin\phi\cos\phi-\dot{\phi}^2]V = 2\dot{\phi}\,\dot{W}+\ddot{\phi}W \quad \dots \quad (1.12)$$

and

$$\ddot{W}+[K_1\sin^2\phi+K_2\cos^2\phi+2L\sin\phi\cos\phi-\dot{\phi}^2]W=-2\dot{\phi}\dot{V}-\ddot{\phi}V \quad \dots \quad (1.13)$$

Since $\phi = \phi'$, we get from (1.12), (1.13) and (19.3), (19.4)

$$V = V_H \quad \text{and} \quad W = W_H$$

i.e.

$$E_x \cos\phi + jH_y \sin\phi = jH_y \cos\phi + H_x \sin\phi$$

$$-E_x \sin\phi + jE_y \cos\phi = -jH_y \sin\phi + H_x \cos\phi \qquad \dots \quad (21)$$

From (21)

$$\frac{H_x}{H_y} = -\frac{E_y}{E_x}$$

APPENDIX

$$\tan\phi_1 = \rho_1 = G - \sqrt{1+G^2} \qquad \dots \quad (i)$$

$$\therefore \quad \cos 2\phi_1 = -\frac{\rho_1^2 - 1}{\rho_1^2 + 1} = \frac{G}{\sqrt{1+G^2}} \qquad \dots \quad (ii)$$

$$\tan\phi_2 = \rho_2 = G + \sqrt{1+G^2} \qquad \dots \quad (iii)$$

$$\cos 2\phi_2 = -\frac{\rho_2^2 - 1}{\rho_2^2 + 1} = -\frac{G}{\sqrt{1+G^2}} \qquad \dots \quad (iv)$$

Now,

$$q^2 = K_1 \cos^2\phi_2 + K_2 \sin^2\phi_2 - 2L \sin\phi_2 \cos\phi_2$$

$$= \frac{K_1 + K_2}{2} + \frac{1}{2}(K_1 - K_2)\cos 2\phi_2 - L \sin 2\phi_2$$

Since, $(K_1 - K_2)/2L = G = -\cot 2\phi_2$

$$q^2 = \frac{K_1}{2}\left(1 + \frac{1}{\cos 2\phi_2}\right) + \frac{K_2}{2}\left(1 - \frac{1}{\cos 2\phi_2}\right)$$

Using (iv), (i) and (iii)

$$q^2 = \frac{K_1\rho_1 + K_2\rho_2}{2G} = \frac{K_1\rho_1 + K_2\rho_2}{\rho_1 + \rho_2}$$

Now putting

$$\left.\begin{aligned} K_1 &= 1 - A_1 \quad \text{where} \quad A_1 = \frac{r}{c'}\,[(\beta'^2 - r\beta') - \omega_x^2] \\ &\text{and} \\ K_2 &= 1 - A_2 \quad \text{where} \quad A_2 = \frac{r}{c'}\,(\beta'^2 - r\beta'] \end{aligned}\right] \qquad \dots \quad (v)$$

We get

$$q^2 = 1 - \frac{A_1\rho_1 + A_2\rho_2}{\rho_1 + \rho_2} \qquad \dots \text{(vi)}$$

We have also :

$$L = \frac{r}{C''}(r - \beta')\omega_z \qquad \dots \text{(vii)}$$

$$C'' = (\beta' - r)(\beta' + \rho_1\omega_z)(\beta' + \rho_2\omega_z) \qquad \dots \text{(viii)}$$

Using (v) and (vii)

$$\rho_1 + \rho_2 = 2G = \frac{K_1 - K_2}{2L} = -\frac{r\omega_x^2}{LC''} = -\frac{\omega_x^2}{\omega_z(r - \beta')} \qquad \dots \text{(ix)}$$

Using (ix), (v), (i), (ii)

$$\frac{A_1\rho_1 + A_2\rho_2}{\rho_1 + \rho_2} = -\frac{L}{\omega_z}(\beta' + \rho_1\omega_z) \qquad \dots \text{(x)}$$

Using (x), (vii), (viii)

$$\frac{A_1\rho_1 + A_2\rho_2}{\rho_1 + \rho_2} = \frac{r}{\beta' + \rho_2\omega_z}$$

Hence

$$q^2 = K_1\cos^2\phi_2 + K_2\sin^2\phi_2 - 2L\sin\phi_2\cos\phi_2$$

$$= 1 - \frac{r}{\beta' + \rho_2\omega_z} = \text{square of the complex Refractive Index}$$

for the O-mode.

This is the same as Eqn. (5.1).

Similarly Eqs. (4.1), (4.2) and (5.2) can be deduced.

From Eqs. (i) and (iii), we get ... (xi)

$$\phi_2 - \phi_1 = \pi/2$$

Hence using (xi), (3.1) and (2.2)

$$W_1 = V_2$$

and using (xi), (3,2) and (2.1)

$$W_2 = -V_1$$

Using (12) and (12.1)

$$\dot{\phi}_1 = \dot{\phi}_2$$

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